

Consistent Quantum Reasoning

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Abstract

Precise rules are developed in order to formalize the reasoning processes involved in standard non-relativistic quantum mechanics, with the help of analogies from classical physics. A classical or quantum description of a mechanical system involves a *framework*, often chosen implicitly, and a *statement* or assertion about the system which is either true or false within the framework with which it is associated. Quantum descriptions are no less “objective” than their classical counterparts, but differ from the latter in the following respects: (i) The framework employs a Hilbert space rather than a classical phase space. (ii) The rules for constructing meaningful statements require that the associated projectors commute with each other and, in the case of time-dependent quantum histories, that consistency conditions be satisfied. (iii) There are incompatible frameworks which cannot be combined, either in constructing descriptions or in making logical inferences about them, even though any one of these frameworks may be used separately for describing a particular physical system.

A new type of “generalized history” is introduced which extends previous proposals by Omnès, and Gell-Mann and Hartle, and a corresponding consistency condition which does not involve density matrices or single out a direction of time. Applications which illustrate the formalism include: measurements of spin, two-slit diffraction, and the emergence of the classical world from a fully quantum description.

I Introduction

Seventy years after non-relativistic quantum mechanics reached what is essentially its present form, the ongoing controversy over its conceptual foundations and the proper interpretation of wave functions, measurements, quantum probabilities, and the like shows no sign of abating. Indeed, modern experiments involving neutron diffraction, ions in traps, and quantum optics seem to cry out for a more satisfactory formulation of basic quantum ideas than was available to Feynman [1] when he remarked, in comparing special relativity and quantum

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theory, that while the former was well understood, “nobody understands quantum mechanics.”

This lack of understanding manifests itself in various ways. Take, for example, the well-known fact that the predictions of standard quantum mechanics violate Bell’s inequality [2]. From this it might seem reasonable to conclude that all derivations of Bell’s inequality contain one or more errors, in the sense that either the premises or the procedures of logical inference entering the argument violate one or more principles of standard quantum mechanics. Had someone in 1964 attempted to publish an inequality violating the predictions of special relativity, the experts would have pointed out precisely where the reasoning went astray as soon as the argument reached the printed page, if not earlier. But this did not happen in the case of Bell’s work. Indeed, much of the subsequent literature on the subject [3] would lead one to believe that standard quantum mechanics is either wrong, or illogical, or unintelligible, or at the very least contains a hidden assumption that the world is non-local, an assumption which completely escaped the physicists who first developed the theory (because they did not understand what they were doing?). This despite the fact that all predictions of standard quantum mechanics seem amply supported by every experimental test designed to uncover some flaw.

As a second example, consider the recent assertion by Englert et al. [4] that Bohmian mechanics [5, 6] makes predictions (or at least retrodictions) which disagree with the results of standard quantum mechanics and also with common sense: detectors designed to detect particles passing through them can actually be triggered by particles which, according to the Bohmian interpretation, never come close to the detector. One might have imagined that this observation would have prompted advocates of Bohmian mechanics to withdraw or modify their claim [7] that this theory reproduces all the results of standard quantum mechanics, especially given an independent verification [8] of the essential correctness of the calculations by Englert et al. Instead, the response [8, 9] has been that standard quantum mechanics, in contrast to Bohmian mechanics, does not provide an adequate theoretical framework for sensibly discussing whether the particle passed through the detector; thus one must take the Bohmian result seriously, as the precise outcome of a well-defined theory, however counterintuitive it may appear to be.

The ultimate goal of the research reported in the present article is to place non-relativistic quantum mechanics on as firm and precise a conceptual foundation as that of special relativity. This is not as ambitious a project as might at first seem to be the case, for two reasons. First, a large part of non-relativistic quantum theory is already in a quite satisfactory state, at least by the somewhat lax standards of theoretical physics. The basic conceptual difficulties are focused in a small (but critical) area, the point at which the mathematical formalism of Schrödinger’s equation, Hilbert space operators, and the like, which by now is quite well understood, is used to generate the probabilities needed to compare theoretical predictions with experimental results. Second, a set of ideas which seem to be adequate for integrating the probabilistic and deterministic aspects of quantum theory into a coherent whole are already in the published literature, although their significance has not been widely appreciated. The first of these ideas is von Neumann’s formulation of quantum theory using closed subspaces of Hilbert space to represent quantum properties [10]—to be carefully distinguished from his theory of measurements [11] and his proposal (together with Birkhoff) of a special quantum logic [12], neither of which are employed in the formulation

of quantum theory presented here. Next comes the concept of a consistent history [13], followed by Omnès' proposal that quantum reasoning must involve separate "logics" [14], and the Gell-Mann and Hartle decoherence functional [15]. To this list the present article adds one new idea, or at least a new word: the *framework* in which a classical or quantum description is embedded, and relative to which its truth must be assessed.

A framework corresponds to a consistent family in the notation of [13], or a "logic" in the notation of [14]. It is closely related (but not identical) to the concept of an "interpretation" in mathematical logic [16]. The key to making sense out of the description of a quantum system, either at a single time or as it develops in time, is to note that such a description cannot be made without adopting, at least implicitly, some framework; that the truth of such a description is relative to its framework; and that reasoning in the quantum domain requires the use of *compatible* frameworks. All of these concepts can be made quite precise, and using them should help clear up (what I regard as) some misunderstandings, and respond to various criticisms of the consistent history program [17, 18, 19].

The present work is very much indebted to Omnès' ideas [14]. A crucial difference is that the notion of "truth" is made to depend explicitly on the framework, following certain classical analogies, Sec. II A, and the example of formal logic. By contrast, Omnès defined "truth" in terms of "facts" which arise in a quasi-classical approximation, Sec. V D. The latter does not seem an entirely satisfactory approach for a fundamental theory of nature, and certain criticisms have been made by d'Espagnat [18], and by Dowker and Kent [19]. Omnès himself [20] agrees that his approach has some problems.

The conceptual structure presented in this paper appears adequate to support the position of Englert et al. [4] (the arguments are not presented here), and to find the mistakes (from the perspective of standard quantum mechanics) in at least some derivations of Bell's inequality; see [21]. There are other derivations of Bell's inequality which rest upon counterfactual reasoning: what *would* have happened *if* something had been different; in addition, many quantum paradoxes involve a counterfactual element. Analyzing these will require an extension of the ideas discussed here; see Sec. VI B.

While its main goal is the clarification of conceptual issues through the introduction of suitably precise rules of reasoning in the quantum domain, this paper also contains, in Sec. IV, some new results on quantum histories and consistency conditions. The representation of histories through the use of projectors on tensor products of copies of the Hilbert space seems to be a new idea, and the notion of a "generalized" history based upon this representation includes, but also goes well beyond, previous proposals by the author [13, 22], Omnès [14], and Gell-Mann and Hartle [15]. The "consistency functional" introduced in the same context generalizes the "decoherence functional" of Gell-Mann and Hartle [15], and results in a formulation of fundamental quantum theory which is (transparently) invariant under reversing the direction of time.

Philosophical issues are not the main topic of this paper. However, its central argument provides a number of details supporting a proposal, found in an earlier publication [23], for interpreting quantum mechanics in a "realistic" manner.

Many of the crucial features which distinguish quantum reasoning from its classical counterpart already arise when one considers a single mechanical system at a single instant of time. Hence Secs. II and III, in which the quantum description of a system at one time is developed with the help of classical analogies, form the heart of this paper. In particular,

discussing both classical and quantum systems from the same point of view helps avoid the error of supposing that the latter somehow involve a subjective element not present in the former, while bringing out the genuine differences between the reasoning processes appropriate in the two situations.

Describing how a quantum system evolves in time turns out to be, formally at least, very similar to describing its properties at a single time if one employs the technical tools in Sec. IV, which allow a history to be represented by a single projector on a tensor product space. The notion of *consistency* has, to be sure, no counterpart in a system at a single time, and discussing it in a precise and general way makes Sec. IV more complicated than the other parts of the paper. The reader who is unfamiliar with the use of consistent histories in quantum mechanics should look elsewhere [13, 14, 22] for a discussion of the physical motivation behind the consistency requirement, and simple examples of its use.

The applications in Sec. V can be understood without the technical machinery of Sec. IV if the reader is willing to accept various results on faith; the proofs are, in any case, not included in this paper. These examples are not new (except, perhaps, for the discussion of two-slit diffraction), but have been chosen to illustrate the formulation of quantum reasoning introduced earlier in the paper. Sec. VI contains a summary of this formulation, together with a list of open questions.

II Classical Reasoning

II A Frameworks and Descriptions

A scientific description of a physical system is, at best, an abstract, symbolic representation of reality, or of what the scientist believes that reality to be; the description is never reality itself. Thus it necessarily embodies certain elements of human choice. Nonetheless, at least in the “classical” world of our everyday experience, it is not unreasonable to claim that such a description is, or at least might be, a faithful or “true” representation of reality. Understanding the process of description in the classical realm will assist us in understanding how elements of choice can enter quantum descriptions without necessarily making them any less “objective” than their classical counterparts.

As a first example, consider representing a three-dimensional object, such as a vase, by means of a two-dimensional drawing which shows a projection of the object on a particular plane. It will be convenient to refer to the choice of projection plane and the various conventions for representing salient features of the object in the drawing as a *framework* \mathcal{F} , and the drawing itself as a *statement*, f , with the pair (\mathcal{F}, f) constituting a *description* of the object. The following are obvious properties of such descriptions:

1. A framework is chosen by the person making the description, and without such a choice no description is possible.
2. Choosing a framework has, by itself, no influence on the object being described; on the other hand it constrains what can correctly be said about the object. (Features which are visible in one projection may well be invisible in another.)

3. A framework by itself is neither true nor false. A statement can be true (i.e., a correct representation of the object) or false, but to determine which it is, one must know the corresponding framework. That is, truth or falsity is relative to the framework.
4. The correctness of a description can be tested experimentally. (For example, the distance between two points on the drawing can be checked by measuring the distance between the corresponding features on the object, provided one knows the projection plane and the scale of the drawing.)
5. An arbitrary collection of descriptions, each of which applies individually to the object, can always be thought of as parts of a collective or composite description of the object.

Apart from the comments in parentheses, *every item in this list, with the sole exception of 5, also applies to a quantum description!* That is, quantum descriptions are no less “objective” or “realistic” than their classical counterparts. However, they are different (reflecting, one might suppose, the peculiar nature of quantum reality) in that it is not always possible to combine various descriptions which might separately apply to a particular object into a single composite description. This feature has no classical analog (that we know of), and hence understanding what it means is essential for consistent quantum reasoning. In particular, one needs an appropriate mathematical formalism, discussed in Secs. III and IV below, giving the precise rules for combining (or not combining) quantum descriptions, and reasoning about them, together with examples, of which there are some in Sec. V, which illustrate the rules.

Each of the five points in the preceding list can be illustrated using as an example the intrinsic angular momentum or spin, measured in units of \hbar , of a spin $1/2$ quantum particle. One possible description, which employs the x component of the spin, has the form:

$$(S_x, 1/2), \tag{1}$$

where S_x is the framework and $1/2$ is the statement. Together they form the description usually written as $S_x = 1/2$. Another equally good description is:

$$(S_x, -1/2), \tag{2}$$

or $S_x = -1/2$. Because (1) and (2) employ the same framework, the same (framework dependent!) notion of “truth” applies to both. In fact, if (1) is true, (2) is false, and *vice versa*. Choosing the framework S_x , i.e., choosing to talk about S_x , does not make $S_x = 1/2$ true or false, in fact, it has no effect whatsoever on the spin of the particle. However, such an assertion is subject to experimental verification (or falsification) “in principle”, that is to say, in terms of idealized experiments which, while they may not be practical, at least do not violate the principles of quantum theory itself. Thus we might imagine that the spin $1/2$ particle is a neutron traveling slowly towards a Stern-Gerlach apparatus whose magnetic field gradient is in the x direction, equipped with a pair of counters to determine the channel in which the neutron emerges. If the neutron later emerges in the $+1/2$ channel, that will verify the correctness of the description $S_x = 1/2$ at the present time, before the neutron enters the field gradient. Even the “classical” problem of checking a two-dimensional projection of a three-dimensional object by making measurements on the object requires a

certain theoretical analysis: relating the distance between two points on the drawing to the measured distance between the corresponding features on the object requires knowing both the projection plane and the scale factor of the drawing, and using some trigonometry. Similarly, experimental verification of a quantum description requires an appropriate theory of measurement, as discussed in Sec. V A.

We now come to item 5 in the list. The description

$$(S_z, -1/2), \tag{3}$$

or $S_z = -1/2$, which uses the z component of the spin, is on a par with (1) or (2). It simply represents a “projection” of the particle’s angular momentum on a different axis. Were we dealing with a classical spinning object there would be not the slightest problem in combining its various components of angular momentum into a complete description corresponding to a three-dimensional vector. But in the quantum case, the frameworks S_x and S_z are *incompatible*, a term which will later receive a precise definition, and as a consequence one of the basic rules of quantum descriptions asserts that *there is no way to combine $S_x = 1/2$ and $S_z = -1/2$ into a single description*. Note that this is *not* the assertion that if $S_x = 1/2$ is “true”, then $S_z = -1/2$ is “false”. Truth and falsity are concepts relative to a particular framework, and in the case of descriptions using S_x and S_z , there is no single framework in which a common notion of truth can be applied to both of them.

The issue of compatibility of frameworks arises because standard quantum mechanics, unlike classical mechanics, employs a Hilbert space, rather than a classical phase space, for descriptions. Incompatibility reflects the fact that certain operators do not commute with each other, a peculiarly quantum phenomenon. In terms of a spin $1/2$ particle, the Hilbert space structure manifests itself in the following way. Descriptions (1), (2), and (3) correspond, as is well known, to particular one-dimensional subspaces or rays of the two-dimensional Hilbert space \mathcal{H} which represents the spin of the particle; in fact, they are the subspaces spanned by the corresponding eigenvectors of the operator S_x or S_z . But as there are no simultaneous eigenvectors of S_x and S_z , there are no subspaces of \mathcal{H} which might be thought of as corresponding to $S_x = 1/2$ and $S_z = -1/2$, that is, to the conjunction of these two descriptions. Another way of thinking about the matter is to note that in classical mechanics, specifying the z component of angular momentum provides additional information beyond that obtained through specifying the x component, whereas for a spin $1/2$ particle one cannot specify any information in addition to $S_x = 1/2$, which is already a pure state.

It is helpful to think of a framework, classical or quantum, as defining a “topic of conversation” and, just as in ordinary conversation, in scientific discourse the framework is often chosen implicitly. The phrase “oranges cost 25 cents each at the supermarket” both defines a topic of conversation and simultaneously asserts something about the nature of the world, and the same is true of “ $S_x = 1/2$ ”. While it is hard to imagine any efficient scheme of scientific communication which did not define frameworks implicitly most of the time, it is sometimes helpful, especially in the quantum case, to reflect upon just what framework is being used at some point in a discussion, as this can help avoid the nonsensical statements and general confusion which arises when an invalid change of framework occurs in the middle of an argument.

II B Classical Phase Space

The usual framework for discussing a classical mechanical system of N particles at a single instant of time is provided by its $6N$ -dimensional phase space D , in which a single point x represents the state of the system at the time in question. A statement p , such as: “the total energy is between 10 and 11 ergs”, can then be thought of as associated with the subset

$$P = \varphi(p) \tag{4}$$

of points in D where p is true, that is, the set contained between two surfaces of constant energy, one at 10 ergs and the other at 11 ergs. The negation of p , denoted $\sim p$, corresponds to $D \setminus P$, the complement of P in D , the region in the phase space for which the energy is less than 10 ergs or greater than 11 ergs. Another statement q : “there are exactly 35 particles in the volume element V_1 ”, where V_1 denotes some definite region in three-dimensional space, corresponds to the subset $Q = \varphi(q)$ of D . The conjunction of p and q , “energy between 10 and 11 ergs *and* 35 particles in V_1 ”, written $p \wedge q$, corresponds to the intersection $P \cap Q$ of the sets P and Q . Thus one sees that various “elementary” statements which describe properties of the system can be mapped onto subsets of D , and other “composite” statements formed from the elementary statements by use of the *logical operations* “not”, “and”, “or”, denoted respectively by \sim, \wedge, \vee , are then also mapped to subsets of D using complements, intersections, and unions of the sets corresponding to the elementary statements. Note that under the operations of complement, union, and intersection, the subsets of D form a Boolean algebra \mathcal{B} .

It will be convenient to think of the *framework* \mathcal{F} as consisting of elementary and composite statements, together with the set D , the Boolean algebra \mathcal{B} of its subsets, and the function φ which maps the statements onto elements of \mathcal{B} . A description (\mathcal{F}, f) , where f is one of the statements, corresponds to the assertion that the state of the mechanical system is one of the points in the subset $F = \varphi(f)$, and is true or false depending upon whether the point x corresponding to the actual state of the system at the time in question is or is not in F . Note that many different statements may correspond to the same F ; for example, $p \vee \sim p$ for any statement p is mapped onto the entire phase space D . When it is not important to distinguish the different possibilities corresponding to some F , one may use a description (\mathcal{F}, F) , with F a surrogate for any statement which is mapped onto it by φ .

A description formed in this way is what in first-order predicate logic [16] is called an *interpretation*. For our purposes it suffices to summarize the rules of logical reasoning, *modus ponens* and the like, as applied to descriptions of our classical mechanical system, in the following way. The logical process of inference from a set of *assumptions* to valid *conclusions* consists in taking the intersection, let us call it A , of all the subsets of D corresponding to all of the assumptions—if one prefers, A is the image under φ of the single statement consisting of the conjunction (“and”) of all the assumptions. Then any conclusion which corresponds to a subset C of D which contains A is a valid conclusion from these assumptions. That is to say, logical reasoning in this context is the process of checking that if there is any x in D for which all the assumptions are true, then for this x the conclusion will also be true. Note that the process of inference can begin with no assumptions at all, in which case one sets A equal to D , and a valid conclusion is a tautology, such as $p \vee \sim p$.

III Quantum Systems at One Time

III A Quantum Frameworks

The counterpart for a quantum system of the classical phase space D is a Hilbert space \mathcal{H} , and an elementary statement p which ascribes to the quantum system a property at a particular time is associated with a closed subspace \mathcal{P} of \mathcal{H} or, equivalently, the *projector* (orthogonal projection operator) P onto this subspace. For example, if p is the assertion that “the total energy is between 10 and 11 ergs”, the subspace \mathcal{P} is spanned by the eigenvectors of \mathcal{H} with eigenvalues (assumed, for simplicity, to be discrete) which lie between 10 and 11 ergs. The negation $\sim p$, which asserts that the energy lies outside this interval, corresponds to the orthogonal complement \mathcal{P}^\perp of \mathcal{P} (not to be confused with $\mathcal{H} \setminus \mathcal{P}$), with projector $I - P$, where I is the identity operator on \mathcal{H} .

A *framework* \mathcal{F} in the quantum case is generated by a finite collection of elementary statements associated with projectors onto closed subspaces of \mathcal{H} *provided these projectors commute with each other*. The projector associated with a statement p will be denoted by $\varphi(p)$. The additional statements belonging to \mathcal{F} are produced from the elementary statements using logical operations, as discussed in Sec. II, and are mapped by φ onto projectors consistent with the rules

$$\varphi(\sim p) = I - \varphi(p), \quad \varphi(p \wedge q) = \varphi(p)\varphi(q). \quad (5)$$

(These rules suffice, because every logical operation can be built up using “not” and “and”.) The smallest family \mathcal{B} of commuting projectors closed under complements and products, and containing the projectors associated with the statements of \mathcal{F} , is a Boolean algebra in which the operations of \cap and \cup , thought of as acting on pairs of projectors, are defined by

$$P \cap Q = PQ, \quad P \cup Q = P + Q - PQ, \quad (6)$$

and whose least and largest elements are \emptyset (the zero operator) and I , respectively. The framework \mathcal{F} can then be defined as the collection of statements generated from a set of elementary statements by logical operations, along with the Hilbert space \mathcal{H} and the mapping φ which carries statements onto the Boolean algebra \mathcal{B} of commuting projectors in the manner indicated above.

A quantum *description* (\mathcal{F}, f) consists of a framework \mathcal{F} and a statement f belonging to the collection of statements associated with \mathcal{F} . Sometimes it is convenient to use a description (\mathcal{F}, F) , where the projector F is an element of the Boolean algebra \mathcal{B} associated with \mathcal{F} , and serves as a surrogate for any f mapped to this F by φ . As long as \mathcal{F} is held fixed, there is a close analogy with classical descriptions based upon a phase space as discussed in Sec. II B above. Since all the projectors in the Boolean algebra \mathcal{B} associated with \mathcal{F} commute with each other, it is possible to choose a representation in which they are simultaneously diagonal, which means that each diagonal element $\langle j|P|j \rangle$ of a projector $P = \varphi(p)$ in \mathcal{B} is either 0 or 1. If one thinks of the set of labels $\{j\}$ of the diagonal elements as constituting a set D , then the subset of D where $\langle j|P|j \rangle = 1$ is analogous to the set of points in the classical phase space D where the statement p is true. This analogy works very well as long as \mathcal{F} is held fixed, and permits one to construct the rules for quantum reasoning in close analogy with their classical counterparts.

In particular, one thinks of a quantum description (\mathcal{F}, p) as *true* provided the quantum system at the time in question is “in” the subspace onto which $P = \varphi(p)$ projects. To be more precise, if we assume that any projector B belonging to the Boolean algebra \mathcal{B} associated with \mathcal{F} represents a statement about, or property of the quantum system which is true at the time in question, then this implies the truth of any statement s with the property that

$$BS = B, \quad (7)$$

where $S = \varphi(s)$ is the projector corresponding to s , that is, provided the subspace onto which B projects is contained in the subspace onto which S projects. Just as in the case of the classical descriptions discussed in Sec. II, the truth of a quantum description must always be thought of as relative to a framework, which may have been defined implicitly. Hence it is best to think of “ (\mathcal{F}, f) is true” as meaning: “Given \mathcal{F} , then f is true.”

Quantum reasoning within the context provided by a single framework \mathcal{F} proceeds in the following way. The assumptions of the argument, a_1, a_2, \dots, a_l , which must all be statements belonging to \mathcal{F} , are mapped by φ onto a set of projectors A_1, A_2, \dots, A_l belonging to \mathcal{B} , whose product is:

$$A = A_1 A_2 \cdots A_l. \quad (8)$$

Then a statement c in \mathcal{F} is a valid conclusion provided $AC = A$, where $C = \varphi(c)$.

III B Compatible and Incompatible Frameworks

The most important differences between classical and quantum descriptions emerge when one considers several different frameworks. In the classical case, as long as the frameworks refer to the same system, there is no problem in combining the corresponding descriptions. But in the quantum case this is no longer true, and it is necessary to pay attention to the rules which state when descriptions can and cannot be combined.

A finite collection of frameworks $\{\mathcal{F}_i\}, i = 1, 2, \dots, l$, will be said to be (mutually) *compatible* if each framework employs the same Hilbert space \mathcal{H} , and if all the projectors associated with the different Boolean algebras \mathcal{B}_i *commute with one another*. In addition, the statements belonging to the different frameworks should be mapped onto the projectors in a consistent way, so that an elementary statement which occurs in more than one framework is mapped to the same projector. (This last point is a matter of notational consistency which causes little difficulty in practice.) Given a compatible collection $\{\mathcal{F}_i\}$ there is a smallest framework \mathcal{F} which contains them all: its statements are generated from the union of the sets of elementary statements for the individual \mathcal{F}_i , and its Boolean algebra \mathcal{B} of projectors is the smallest one containing all the projectors of all the Boolean algebras \mathcal{B}_i associated with the different frameworks in the collection. We shall say that the compatible collection $\{\mathcal{F}_i\}$ *generates* this smallest framework \mathcal{F} .

Two or more frameworks which are not compatible are called *incompatible*. The distinctive problems associated with quantum reasoning arise from the existence of frameworks which use the same Hilbert space, and can thus (potentially) refer to the same physical system, but which are mutually incompatible because the projectors associated with one framework do not commute with those of another. There is nothing quite like this in classical mechanics, since the classical counterparts of projectors always commute with one

another.

We adopt the following as a fundamental rule. *A meaningful quantum description must consist of a single framework and one of its statements, (\mathcal{F}, f) , or else a compatible collection of frameworks $\{\mathcal{F}_i\}$ and associated statements $\{f_i\}$, which together form a collective description $\{(\mathcal{F}_i, f_i)\}$.* A collective description can always be replaced by a single *master description* (\mathcal{D}, d) , where \mathcal{D} is the framework generated by the collection $\{\mathcal{F}_i\}$, and d is any statement such that

$$D = \varphi(d) = F_1 F_2 \cdots F_l \quad (9)$$

is the product of the projectors $F_i = \varphi(f_i)$ corresponding to the different statements.

The rules for logical reasoning about quantum descriptions are similar to the rules for reasoning about classical descriptions, but with the additional requirement that all the frameworks must be compatible. A logical *argument* begins with a set of *assumptions*, which is to say a set of descriptions $\{(\mathcal{A}_i, a_i)\}$, $i = 1, 2, \dots, l$, associated with a *compatible* family $\{\mathcal{A}_i\}$ of frameworks; one is assuming that these descriptions are simultaneously true (within the framework generated by the collection $\{\mathcal{A}_i\}$). From these assumptions one can deduce a set of valid *conclusions* $\{(\mathcal{C}_j, c_j)\}$, $j = 1, 2, \dots, m$, provided the union of the collections $\{\mathcal{A}_i\}$ and $\{\mathcal{C}_j\}$ is a compatible collection of frameworks, and

$$\varphi(c_j)A = A, \quad (10)$$

holds for every j , where

$$A = A_1 A_2 \cdots A_l \quad (11)$$

is the product of the projectors $A_i = \varphi(a_i)$. The set of assumptions can always be replaced by the corresponding master description acting as a single assumption, since the process of reasoning just described allows one to deduce the original set of assumptions from this master description. A valid argument can begin with the true assumption (\mathcal{F}, I) , where \mathcal{F} is any framework, and I is the identity operator.

It is extremely important that within a *single* argument, *all* the frameworks, for both the assumptions and conclusions, be *compatible*. In particular, the following sort of “quasi-classical reasoning”, many examples of which are to be found in the published literature, is not valid: One starts with an assumption (\mathcal{A}, a) , and from it deduces a conclusion (\mathcal{C}, c) , after checking that the frameworks \mathcal{A} and \mathcal{C} are compatible. Next, (\mathcal{C}, c) is used as the assumption in an argument whose conclusion is (\mathcal{E}, e) , again after checking that the frameworks \mathcal{C} and \mathcal{E} are compatible. Combining these two arguments one draws the conclusion that “*is a is true, then e must be true*”. But this reasoning process is only correct if \mathcal{A} and \mathcal{E} are compatible frameworks; if they are incompatible, it is invalid. (Note that compatibility is not a transitive relationship: \mathcal{A} can be compatible with \mathcal{C} and \mathcal{C} with \mathcal{E} at the same time that \mathcal{A} is incompatible with \mathcal{E} .) Failure to check compatibility can easily lead to inconsistent quantum reasoning which, just like inconsistent classical reasoning, often produces contradictions and paradoxes. In order to avoid making mistakes of the sort just described, it is helpful to think of a single quantum argument as a process in which the assumptions, which themselves form a (collective) description, are extended into a longer and longer collective description by adding additional simple descriptions without ever erasing anything; in particular, without forgetting the original assumptions. The compatibility rule for collective quantum descriptions will then rule out any attempt to introduce incompatible frameworks.

On the other hand, two or more *separate* arguments may very well involve different, and perhaps incompatible frameworks. But then they cannot be combined into a single argument. As an example, it is possible to construct *two* perfectly valid arguments, each based upon the *same* assumption (\mathcal{A}, a) , one leading to the conclusion (\mathcal{C}, c) and the other to the conclusion (\mathcal{E}, e) , where the frameworks \mathcal{C} and \mathcal{E} are incompatible. If one grants the truth of the assumption, then c is true relative to framework \mathcal{C} , and e is true relative to framework \mathcal{E} . However, there is no framework relative to which it can be said that both c and e are true statements. Given a particular physical system for which the assumptions are satisfied, it is possible in principle to check the validity of either c or e by means of measurements, but not (at least in general) the validity of both. An example (involving histories) which illustrates this point will be found in Sec. V B below.

If one wishes to describe a quantum system, it is necessary to choose a framework, if only implicitly, and in all but the most trivial cases this choice means that there are many incompatible frameworks whose statements cannot be employed in the description. While the choice of a framework does not in any way “influence” the system being described, Sec. II A, it severely constrains what can sensibly be said about it. Thus if one wants to talk about the energy, this requires the use of certain projectors, depending on how precisely one wants to specify the energy, and what ranges one is interested in. It is then not possible, as part of the same description, to talk about something else represented by projectors which do not commute with the set employed for the energy.

The type of exclusion which arises from incompatible frameworks is easily confused with, but is in fact quite different from the sort of exclusion which arises all the time in classical mechanical descriptions, where in order that some property p be true, it is necessary that some other property q be false (because the corresponding subsets of the classical phase space do not overlap, Sec. II B). This “classical” type of exclusion also arises in quantum systems where, to take the example of a spin $1/2$ particle, “ $S_x = 1/2$ ” and “ $S_x = -1/2$ ” are mutually exclusive statements belonging to the same framework: if one is true, the other is false.

By contrast, if p and q are assertions about a particular quantum system represented by projectors P and Q which do not commute, they cannot be part of the same framework. Hence in a framework \mathcal{P} in which it makes sense to talk about the truth and falsity of p , there is no way of discussing whether q is true or false, since within this framework q makes no sense. Conversely, in a framework \mathcal{Q} in which q makes sense, and could be true or false, p has no meaning. Thus the truth of p does not make q false; it does mean that because we have (perhaps implicitly) adopted a framework in which it makes sense to talk about p , q cannot be discussed. Similarly, the combination $p \wedge q$, or “ p and q ”, is not part of any framework, and therefore cannot be true or false. It is “meaningless” in the sense that within quantum theory one cannot ascribe any meaning to it. In mathematical logic there are certain combinations of symbols, for example $p \wedge \vee q$, which are “nonsense” because they are not “well-formed formulas”, they are not constructed according to the rules for combining symbols of the language to form meaningful statements. In the quantum case, where the rules are, of course, different from those of classical logic, $p \wedge q$ (and also $p \vee q$, etc.) has this nonsensical character when $PQ \neq QP$.

IV Quantum Systems at Many Times

IV A Classical Analogy

A classical stochastic process, such as rolling a die several times in succession, is a more useful analogy to quantum time dependence than is a continuous trajectory in a classical phase space. If a die is rolled three times in succession, the possible outcomes are triples of numbers (d_1, d_2, d_3) , $1 \leq d_j \leq 6$, which together constitute the set

$$\tilde{D} = D^3 \quad (12)$$

of $216 = 6^3$ histories, the sample space of classical probability theory. Here D is the set of possible outcomes of one toss. A statement such as p : “the sum of the first two tosses is four”, is associated with the subset $\tilde{P} = \varphi(p)$ of \tilde{D} consisting of those histories for which p is true. The process of logical inference employing statements of this kind is then formally identical to that discussed in Sec. II.B. For example, from p one can immediately infer r : “the sum of all three tosses is five or more” by noting that \tilde{P} is a subset of $\tilde{R} = \varphi(r)$, the set of histories where r is true.

On the other hand, the statement q : “the sum of all three tosses is exactly seven”, is not a logical consequence of p , since $\tilde{Q} = \varphi(q)$ is a subset of p . However, if probabilities are assigned to \tilde{D} , one can compute the *conditional probability*

$$\Pr(q|p) = W(\tilde{Q} \cap \tilde{P})/W(\tilde{P}) \quad (13)$$

of q given p , using *weights* $W(\tilde{A})$ for subsets \tilde{A} of \tilde{D} . In the case of an unloaded die, $W(\tilde{A})$ is just the number of elements (histories) in the set \tilde{A} . On the other hand, making appropriate replacements in (13), one sees that $\Pr(r|p) = 1$. Indeed, as long as every history has a finite weight, there is a close connection between r being a *logical* consequence of p , and $\Pr(r|p) = 1$; the only difference comes about in the case in which the statement p corresponds to an empty set with zero weight, for which the right side of (13) is undefined.

IV B Histories and Projectors

The simplest type of *history* for a quantum system with Hilbert space \mathcal{H} is constructed in the following way. Let a finite set of times $t'_1 < t'_2 < \dots < t'_j$ be given, and at each t'_j let P'_j be a projector onto a closed subspace of \mathcal{H} . The history consists of the set of times and the associated projectors, representing properties of the system which are true at the corresponding times. So that this history can be discussed within the same framework as other histories defined at different sets of times, it is convenient to suppose that there is a common set of times

$$t_1 < t_2 < \dots < t_n \quad (14)$$

sufficiently large to include all those associated with the different histories in the framework, and that a particular history is represented by a sequence of projectors P_1, P_2, \dots, P_n , one for each of the times in (14). To do this, P_k is set equal to the identity I if t_k is not one of the times for which the history was originally defined, and to P'_j if $t_k = t'_j$. Since the “property” represented by I is always true, it may be “added” to the original history at additional times without modifying its physical significance.

Rather than representing a history with a sequence of projectors P_1, P_2, \dots, P_n , it is technically more convenient to employ a *single* projector

$$\tilde{P} = P_1 \otimes P_2 \otimes \dots \otimes P_n \quad (15)$$

on the tensor product space

$$\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H} \quad (16)$$

of n copies of \mathcal{H} , which is analogous to \tilde{D} in the classical case discussed earlier. It is obvious that if each of the P_j on the right side of (15) is a projector, so is \tilde{P} ; on the other hand, not every projector on $\tilde{\mathcal{H}}$ has the form (15). It will be convenient to extend the concept of a history to include *any* projector on $\tilde{\mathcal{H}}$. The resulting category of *generalized histories* is large enough to include the proposals made by Omnès [14] (“non-Griffiths histories”), and by Gell-Mann and Hartle [15] (“history-dependent decompositions of the identity”), and much else besides. However, the applications discussed in Sec. V are all of the restricted form (15), hereafter referred to as *simple histories*.

A *framework* \mathcal{F} for the case of many times can be constructed in the following way. We suppose that there are a finite number of elementary statements p, q, \dots , and that to each of these there corresponds a projector on $\tilde{\mathcal{H}}$, representing some (generalized) history. As in the case of a quantum system at one time, we require that all these projectors commute with each other. They are therefore members of a smallest Boolean algebra $\tilde{\mathcal{B}}$ of commuting projectors closed under the operations of products and complements, where $\tilde{I} - \tilde{P}$ is the complement of \tilde{P} , and

$$\tilde{I} = I \otimes I \otimes \dots \otimes I \quad (17)$$

is the identity on $\tilde{\mathcal{H}}$. Additional statements formed from the elementary statements by logical operations will also be mapped onto elements of $\tilde{\mathcal{B}}$ following the rules in (5). Thus the structure is formally the same as that in Sec. III. However, for a framework involving histories to be acceptable it must also satisfy consistency conditions, which is the next topic.

IV C Consistency and Weights

Various consistency conditions have been proposed by various authors; what follows is closest in spirit to [22], while very much indebted to the ideas in [15]. In the following discussion it will be convenient to assume that the Hilbert space \mathcal{H} is *finite dimensional*, so that there are no questions about convergence of sums. This will not bother low-brow physicists who are quite content to consider a quantum system in a box of finite volume with an upper bound (as large as one wishes) on the energy. Extending the formalism in a mathematically precise way to an infinite-dimensional \mathcal{H} remains an open problem.

Let $T(t', t)$ be the unitary time transformation which represents Schrödinger time evolution from time t to time t' . In the case of a time-independent Hamiltonian H it is given by:

$$T(t', t) = \exp[-i(t' - t)H/\hbar]. \quad (18)$$

The discussion which follows is not restricted to time-independent Hamiltonians, but we do require that $T(t', t)$ be unitary and satisfy the conditions

$$\begin{aligned} T(t, t) &= I, \\ T(t, t')T(t', t'') &= T(t, t''), \end{aligned} \quad (19)$$

which imply that $T(t, t')$ is the inverse of $T(t', t)$.

Given $T(t', t)$, one can define a mapping from an arbitrary operator \tilde{A} on \mathcal{H}^n to an operator $K(\tilde{A})$ on \mathcal{H} by means of the formula:

$$\begin{aligned} \langle i|K(\tilde{A})|j\rangle &= \sum_{k_2} \sum_{k_3} \dots \sum_{k_n} \sum_{l_1} \sum_{l_2} \dots \sum_{l_{n-1}} \langle ik_2k_3\dots k_n|\tilde{A}|l_1l_2\dots l_{n-1}j\rangle \\ &\times \langle l_1|T(t_1, t_2)|k_2\rangle \langle l_2|(T(t_2, t_3)|k_3)\dots \langle l_{n-1}|T(t_{n-1}, t_n)|k_n\rangle, \end{aligned} \quad (20)$$

where the matrix elements refer to some orthonormal basis $\{|j\rangle\}$ of \mathcal{H} and the corresponding tensor product basis of \mathcal{H}^n . If, in particular, \tilde{A} is a projector \tilde{P} of the form (15), then $K(\tilde{P})$ takes the form:

$$K(\tilde{P}) = P_1T(t_1, t_2)P_2T(t_2, t_3)\dots T(t_{n-1}, t_n)P_n. \quad (21)$$

Using K , we define the bilinear *consistency functional* C on pairs of operators \tilde{A}, \tilde{B} on $\tilde{\mathcal{H}}$ by means of the formula:

$$C(\tilde{A}, \tilde{B}) = C^*(\tilde{B}, \tilde{A}) = \text{Tr}[K^\dagger(\tilde{A})K(\tilde{B})]. \quad (22)$$

Note that $C(\tilde{A}, \tilde{A})$ is non-negative. If both \tilde{A} and \tilde{B} are projectors of the form (15), the consistency functional is the same as the Gell-Mann and Hartle decoherence functional if the density matrix in the latter is replaced by I .

Because $\tilde{\mathcal{H}}$ is finite-dimensional, the Boolean algebra $\tilde{\mathcal{B}}$ associated with \mathcal{F} contains a finite number of projectors, and consequently there are a set of non-zero *minimal elements* $\{\tilde{M}^{(\alpha)}\}$, $\alpha = 1, 2, \dots$ which form a decomposition of the identity,

$$\tilde{I} = \sum_{\alpha} \tilde{M}^{(\alpha)}, \quad \tilde{M}^{(\alpha)}\tilde{M}^{(\beta)} = \delta_{\alpha\beta}\tilde{M}^{(\alpha)}, \quad (23)$$

in terms of which any projector \tilde{P} in $\tilde{\mathcal{B}}$ can be written in the form

$$\tilde{P} = \sum_{\alpha} m_{\alpha}\tilde{M}^{(\alpha)}, \quad (24)$$

with each m_{α} equal to 0 or 1, depending on \tilde{P} . An acceptable framework \mathcal{F} must satisfy the following *consistency condition* in terms of these minimal elements:

$$C(\tilde{M}^{(\alpha)}, \tilde{M}^{(\beta)}) = 0 \text{ whenever } \alpha \neq \beta. \quad (25)$$

If this is satisfied, a non-negative *weight* $W(\tilde{P})$ can be defined for every element of $\tilde{\mathcal{B}}$ through the formula:

$$W(\tilde{P}) = C(\tilde{P}, \tilde{P}), \quad (26)$$

and these weights can be used to generate conditional probabilities in analogy with the classical case, (13), and as discussed in [13] and [22]. In particular, with $\tilde{P} = \varphi(p)$ and $\tilde{Q} = \varphi(q)$ the projectors associated with p and q , the counterpart of (13) is the formula

$$\text{Pr}(q|p) = W(\tilde{P}\tilde{Q})/W(\tilde{P}), \quad (27)$$

with W defined by (26).

If only a single time is involved, then it is easy to show that the consistency condition (25) is automatically satisfied. It is also automatically satisfied for all families of *simple* histories involving only two times, $n = 2$. However, in the case of generalized histories involving two or more times, and simple histories involving three or more times, the consistency condition is a non-trivial restriction on acceptable families.

The consistency functional (22), unlike the decoherence functional of Gell-Mann and Hartle, does not involve a density matrix. This means that the former, unlike the latter, does not single out a direction of time. There seems to be no good reason why a density matrix should appear in a fundamental formulation of quantum mechanics, especially since its classical analog is properly part of classical statistical mechanics rather than classical mechanics as such. To be sure, a density matrix may also arise in quantum mechanics, unlike classical mechanics, as a technical device for describing the state of a subsystem of a total system when the latter is in a pure state. But it is not needed for describing a single closed system, which is what we are considering here. (Also see the remarks in Sec. V D.)

IV D Frameworks, Compatibility, and Logical Inference

The discussion above can be summarized by saying that a framework \mathcal{F} for a quantum system at several times, usually called a *consistent family of histories*, consists of a collection of statements generated from a set of elementary statements by logical operations, together with a mapping φ , conforming in an appropriate way to the logical operations, of these statements onto a Boolean algebra of commuting projectors on $\tilde{\mathcal{H}}$, with the additional requirement that the minimal elements of this algebra satisfy the consistency condition (25).

A *description* is then a pair (\mathcal{F}, f) , consisting of the framework or consistent family \mathcal{F} and one of its statements f . We shall, with a certain lack of precision, call both the description (\mathcal{F}, f) and the corresponding statement f a quantum *history*. This ought not to cause confusion if one remembers that statements by themselves do not have a meaning (and, in particular, cannot be true or false) unless they are embedded in or associated with some framework, which can either be specified explicitly by giving the pair (\mathcal{F}, f) , or implicitly. In particular, if \tilde{F} is the projector corresponding to f , then the smallest Boolean algebra containing \tilde{F} defines, in the absence of any other information, an implicit framework for f , and this algebra will be part of the Boolean algebra of any consistent family which has f as one of its statements. (Note that for a particular projector \tilde{F} , even this smallest Boolean algebra may not satisfy the consistency condition (25); in such a case f is an inconsistent (meaningless) history, which cannot be part of any consistent family.)

The intuitive significance of a history (\mathcal{F}, f) is very similar to that of a description at a single time, as discussed in Sec. III. In particular, in the case of a simple history corresponding to (15), one thinks of the quantum system as actually possessing the property (represented by) P_j at each time t_j , if this history is the one which actually occurs. Quantum mechanics, as a stochastic theory, cannot (in general) guarantee that such a history takes place; instead, it assigns it a probability, based on some assumption, such as the occurrence of the initial state, using the weights generated by the consistency functional. An empirical check on these probabilities is possible “in principle”, i.e., by idealized measurements which do not violate the principles of quantum theory; see the extensive discussion in [13].

For a finite collection $\{\mathcal{F}_i\}$ of consistent families to be (mutually) *compatible*, they must,

to begin with, refer to a common tensor product $\tilde{\mathcal{H}}$, (16), or else it must be possible to achieve this by defining $\tilde{\mathcal{H}}$ using a larger collection of times than those found in the individual families, and then extending the latter by adding identity operators to the histories at the times not previously considered. Next comes a requirement of notational consistency: statements belonging to the different frameworks must be mapped onto the projectors in a consistent way, so that an elementary statement which occurs in more than one framework is mapped to the same projector. There are then two additional and specifically quantum conditions. The first is the same as in Sec. III.A: projectors corresponding to the different Boolean algebras $\tilde{\mathcal{B}}_i$ of the different families must commute with each other. The second is the consistency condition (25) imposed on the minimal elements of the Boolean algebra $\tilde{\mathcal{B}}$ associated with the smallest consistent family \mathcal{F} containing all the \mathcal{F}_i , the family generated by the collection $\{\mathcal{F}_i\}$.

Given these definitions, one has the same fundamental rule as in Sec. III.B; worded in terms of histories, it reads: *A meaningful quantum history must consist of a single consistent family together with one of its histories (statements), (\mathcal{F}, f) , or else a compatible collection of consistent families $\{\mathcal{F}_i\}$ and associated histories $\{f_i\}$, which together form a collective history (or description) $\{(\mathcal{F}_i, f_i)\}$.* A collective history can always be replaced by a single *master history* (\mathcal{D}, d) , where \mathcal{D} is the consistent family generated by the collection $\{\mathcal{F}_i\}$, and d is any history corresponding to the projector \tilde{D} which is the product of the projectors \tilde{F}_i corresponding to the histories f_i , as in (9).

The rules for logical reasoning in the case of histories are similar to those for a quantum system at one time, Sec. III B, except that they are based upon conditional probabilities constructed from ratios of weights, using (27). An *argument* begins with a set of *assumptions*, which is to say a set of histories or descriptions $\{(\mathcal{A}_i, a_i)\}, i = 1, 2, \dots, l$, associated with a *compatible* set $\{\mathcal{A}_i\}$ of consistent families; one is assuming that these histories are simultaneously true (within the consistent family generated by the collection $\{\mathcal{A}_i\}$). From these one can deduce a set of valid *conclusions* $\{(\mathcal{C}_j, c_j)\}, j = 1, 2, \dots, m$, provided the union of the collections $\{\mathcal{A}_i\}$ and $\{\mathcal{C}_j\}$ is a compatible collection of consistent families, and provided

$$\Pr(c_j | \tilde{A}) = W(\tilde{C}_j \tilde{A}) / W(\tilde{A}) = 1 \quad (28)$$

holds for every j , where $\tilde{C}_j = \varphi(c_j)$, and

$$\tilde{A} = \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_l, \quad (29)$$

with \tilde{A}_i the projector associated with the history a_i . In order for the inference to be valid, we require that $W(\tilde{A})$ be positive, so the right side of (28) is defined. The case $W(\tilde{A}) = 0$ corresponds to a probability of zero that all of the assumptions in the argument are simultaneously satisfied, and is thus similar to having contradictory hypotheses in ordinary logic. Excluding the possibility of making any inferences from probability zero cases, as suggested here, seems intrinsically no worse than the solution in ordinary logic, in which any statement whatsoever can be inferred from a contradiction.

These rules coincide with those given in Sec. III B in the particular case where $n = 1$, assuming \mathcal{H} is finite dimensional, because $W(P)$ for a projector P is then just the dimension of the space onto which P projects. The case of contradictory assumptions, $A = 0$, again forms an exception, for the reasons discussed above.

The remarks at the end of Sec. III B about the necessity of choosing a particular framework in order to describe a quantum system also apply to a quantum history. It is necessary to choose some consistent family, if only implicitly, in order to describe the time development of a quantum system, and while this choice has no influence upon the system itself, it constrains what can sensibly be said about it. The consistent family is not itself either true or false, but a history can be true or false, i.e., occur or not occur, as one of the possibilities within a consistent family. As long as one considers a single family, a particular history may well exclude another history in the sense that the probability is zero that both occur. But this sort of exclusion is very different from that which arises when two histories f and g are incompatible in the sense that there is no consistent family which includes both of them. In such a case, the occurrence of history f does *not* imply that g does not occur; what it means is that in order to even talk about the occurrence of f , we must employ a framework or consistent family in which it makes no sense to say whether or not g occurs, since g is not one of the histories in this family, and could not be added to this family without making it inconsistent. For the same reason, statements such as “ f and g ”, or “ f or g ”, make no sense when these histories cannot both belong to a single consistent family. Such combinations are like improperly-formed formulas in mathematical logic, sequences of symbols which cannot be interpreted because they are not constructed according to the rules appropriate to the particular language under discussion.

V Applications

V A Measurements

As a simple example of a measurement, consider the process of determining S_x , the x component of spin, for a spin 1/2 particle using a Stern-Gerlach apparatus equipped with detectors to determine in which channel the particle emerges from the region where there is a magnetic field gradient. The essential features of the unitary time transformation $T(t_2, t_1)$ from a time t_1 before the measurement takes place to a time t_2 after it is completed are, in an obvious notation:

$$|\alpha, X\rangle \rightarrow |\alpha', X^+\rangle, \quad |\beta, X\rangle \rightarrow |\beta', X^-\rangle, \quad (30)$$

where $|\alpha\rangle$ corresponds to the spin state $S_x = 1/2$ for the particle, in units of \hbar , $|\beta\rangle$ to $S_x = -1/2$, $|X\rangle$ to the “ready” state of the apparatus before the particle has arrived, and $|X^+\rangle$ and $|X^-\rangle$ are apparatus states which result if the particle emerges from the field gradient in the $S_x = 1/2$ or $S_x = -1/2$ channel, respectively. One should think of $|X^+\rangle$ and $|X^-\rangle$ as macroscopically distinct states corresponding to, say, two distinct positions of a pointer which indicate the outcome of the measurement. The spin states $|\alpha'\rangle$ and $|\beta'\rangle$ at time t_2 are arbitrary; they are not relevant for the measuring process we are interested in. Throughout the following discussion we employ a symbol outside a Dirac ket to indicate the corresponding projector; for example:

$$\alpha = |\alpha\rangle\langle\alpha|, \quad X^+ = |X^+\rangle\langle X^+|. \quad (31)$$

As a first framework or consistent family \mathcal{F}_1 , we use the one generated by α , β and X at t_1 , and X^+ and X^- at t_2 . To use the terminology of Sec IV, \mathcal{F}_1 contains statements

of the type p : “ $S_x = 1/2$ at time t_1 ”, q : “the pointer at t_2 corresponds to the apparatus having detected the particle emerging in the $S_x = 1/2$ channel”, and the like, which are then mapped to projectors

$$\varphi(p) = \alpha \otimes I, \quad \varphi(q) = I \otimes X^+, \quad (32)$$

etc., on \mathcal{H}^2 . The result is a Boolean algebra of projectors associated with a family of simple histories at only two times, which is therefore automatically consistent.

Given the framework \mathcal{F}_1 , it is possible to use the weights (26) generated by the consistency functional, see [13, 22], to calculate various conditional probabilities such as:

$$\Pr(\alpha, t_1 | X^+, t_2) = 1, \quad \Pr(\beta, t_1 | X^+, t_2) = 0. \quad (33)$$

Stated in words, given that the apparatus is in state X^+ at t_2 , one can be sure that the particle was in the spin state $S_x = 1/2$ and not $S_x = -1/2$ at time t_1 . This is certainly the sort of result which one would expect to emerge from a reasonable quantum theory of measurement, and which does, indeed, emerge if one chooses an appropriate framework.

Next, consider a situation in which for some reason one knows that the particle at time t_1 has a spin polarization $S_z = 1/2$ corresponding to the spin state

$$|\gamma\rangle = (|\alpha\rangle + |\beta\rangle)/\sqrt{2}; \quad (34)$$

for example, the particle may have come through a spin polarizer which selected this polarization. What will happen during the measurement process? We adopt as a framework the consistent family \mathcal{F}_2 which is generated by γ ($S_z = 1/2$),

$$|\delta\rangle = (|\alpha\rangle - |\beta\rangle)/\sqrt{2} \quad (35)$$

($S_z = -1/2$), and X at t_1 ; and, as before, X^+ and X^- at t_2 . Using this framework (again, consistency is automatic) and the weights generated by the consistency functional, one can calculate probabilities such as:

$$\Pr(X^+, t_2 | \gamma, X, t_1) = \Pr(X^-, t_2 | \gamma, X, t_1) = 1/2. \quad (36)$$

Stated in words, given that $S_z = 1/2$ and the apparatus was ready at t_1 , the probability is $1/2$ that the apparatus is in the X^+ state, and $1/2$ that it is in the X^- state at time t_2 .

All discussions within the framework of standard quantum mechanics eventually arrive at the conclusion (36), but many of them are forced to make equivocations and unsatisfactory excuses along the way. The reason is that they adopt (implicitly) yet another framework, \mathcal{F}_3 , which is generated by γ and δ at t_1 , and at t_2 the state

$$|G\rangle = T(t_2, t_1)|\gamma, X\rangle = (|\alpha', X^+\rangle + |\beta', X^-\rangle)/\sqrt{2}. \quad (37)$$

with projector G . To be sure, \mathcal{F}_3 is an acceptable framework, the consistency conditions are satisfied, and within this framework one can derive the conditional probability

$$\Pr(G, t_2 | \gamma, X, t_1) = 1. \quad (38)$$

Stated in words, it is the case that *if one uses the consistent family \mathcal{F}_3* , the state G will occur with certainty at t_2 , given that $S_z = 1/2$ and the apparatus was ready at t_1 .

The problem, of course, is that $|G\rangle$ is a macroscopic quantum superposition (MQS), or Schrödinger’s cat state, and tells one nothing whatsoever about the position of the pointer on the apparatus. Indeed, adopting \mathcal{F}_3 makes it impossible even to discuss whether the apparatus is in state X^+ or X^- at the end of the measuring process, because the projectors X^+ and X^- do not commute with G . All attempts to somehow deduce that the pointer is in one position or the other “for all practical purposes” are nonsensical once framework \mathcal{F}_3 has been adopted, in agreement with Bell’s observations [24], based upon more intuitive, but nonetheless quite reasonable arguments. If one wants to talk about the pointer positions, it is necessary to adopt a framework which includes the possibility of making references to such positions, for example, \mathcal{F}_2 . But note that within \mathcal{F}_2 it is equally nonsensical to refer to the MQS state G . Thus one sees that the enormous confusion which surrounds most discussions of “the measurement problem” is generated by a failure to distinguish the different frameworks or consistent families which are possible in quantum theory, and to note that a description employing one framework necessarily excludes certain statements valid in other, incompatible frameworks.

The consistent family \mathcal{F}_2 , while it evades the problem of “ghostly” MQS states, can be faulted as a description of a *measurement* process in that the outcome of the measurement in terms of pointer positions is not correlated with a property of the particle before the measurement. This can be remedied by embedding \mathcal{F}_2 in a larger framework \mathcal{F}_4 which is generated by the events already included in \mathcal{F}_2 at t_1 and t_2 , and, in addition, α and β , $S_x = \pm 1/2$, at a time $t_{1.5}$ which is later than t_1 but earlier than the instant when the particle actually enters the magnetic field gradient of the apparatus. It turns out that \mathcal{F}_4 is consistent (this must be checked, as it is no longer automatic), and using it one can show, among other things, that

$$\Pr(\alpha, t_{1.5} | \gamma, X, t_1; X^+, t_2) = 1, \quad (39)$$

i.e., given the initial state at t_1 , and the fact that at t_2 the pointer indicates that the particle has emerged in the $S_x = 1/2$ channel, it follows that the spin state was $S_x = 1/2$ at the time $t_{1.5}$. Thus, using this framework, one can again say that the apparatus after the measurement indicates a state possessed by the particle before the measurement. There is, incidentally, nothing incompatible between having $S_z = 1/2$ at t_1 and $S_x = 1/2$ at $t_{1.5}$ for the same particle, since these statements belong to the same framework or consistent family. This is just one of the ways in which consistent quantum reasoning, with its precise rules, allows one to go well beyond what is possible in standard quantum mechanics slavishly interpreted in terms of ill-defined “measurements”.

This set of examples shows that a satisfactory theory of quantum measurement requires the use of a framework which includes the possibility of discussing *both* the outcome shown by the apparatus *and*, at a time before the measurement, the properties of the measured system which the measurement is designed to detect. To be sure, precisely the same conditions apply to a satisfactory theory of measurement in the context of classical mechanics. The difference is that in the classical case the choice of an appropriate framework can be made implicitly with no difficulty, whereas in the quantum case it is necessary to choose it with some care, in order to avoid meaningless statements associated with attempts to mix descriptions belonging to incompatible frameworks.

V B Spin of a Particle Between Two Measurements

The following example, with trivial differences in notation, is from [13]; applying the rules for descriptions contained in the present paper clarifies the original presentation and responds to certain criticisms [18, 25].

Consider a spin 1/2 particle which passes through two successive devices which measure the spin polarization, Fig. 1. The first device measures S_x without changing it, so the unitary transformation is

$$|\alpha, X\rangle \rightarrow |\alpha, X^+\rangle, \quad |\beta, X\rangle \rightarrow |\beta, X^-\rangle, \quad (40)$$

which is identical to (30) except that the spin states α ($S_x = 1/2$) and β ($S_x = -1/2$) are the same before and after the measurement. As in Sec. V A, $|X\rangle$ is the “ready” state of the device, and $|X^+\rangle$ and $|X^-\rangle$ correspond to the two different pointer positions indicating the results of the measurement. The second device, Fig. 1, is similar, except that it measures S_z , with states $|Z\rangle$ (“ready”), and $|Z^+\rangle$ and $|Z^-\rangle$ corresponding to having measured $S_z = 1/2$ and $S_z = -1/2$, respectively.

As indicated in Fig. 1, let t_1 be some time before the particle enters the first device, when its spin state is $|\gamma\rangle$, corresponding to $S_z = 1/2$; t_2 a time when it is between the two devices; and t_3 a time when the particle has left the second device, and the pointers on both devices indicate the results of the respective measurements. Suppose that at t_3 the measuring devices are in states $|X^+\rangle$ and $|Z^+\rangle$. What can one conclude about the spin of the particle at the time t_2 when it was between the two devices?

First consider a consistent family \mathcal{F}_1 generated by the initial state

$$|\psi_1\rangle = |\gamma X Z\rangle \quad (41)$$

at t_1 together with the pointer positions for both devices at t_3 . Consistency is automatic, as simple histories at only two times are involved. Using this family, one can calculate the conditional probabilities for the final pointer positions given the initial state; for example,

$$\Pr(X^+ Z^+, t_3 | \psi_1, t_1) = 1/4. \quad (42)$$

Next consider the family \mathcal{F}_2 obtained from \mathcal{F}_1 by adding S_x at time t_2 . This family is consistent, and using the corresponding weights one can show that

$$\Pr(S_x = 1/2, t_2 | \psi_1, t_1; X^+ Z^+, t_3) = 1; \quad (43)$$

for the detailed calculation, see [13]. That is, given the initial state and the final pointer positions, one can be certain that at the intermediate time t_2 the particle was in a spin state $S_x = 1/2$. Note that this inference can be checked, in principle, by inserting a third device between the first two, shown dashed in Fig. 1, which measures S_x , and verifying that it yields the same result as the first device.

One can also consider the family \mathcal{F}_3 obtained from \mathcal{F}_1 by adding S_z at time t_2 . This, too, is consistent, and with the help of the corresponding weights (again, details are in [13]) one can show that

$$\Pr(S_z = 1/2, t_2 | \psi_1, t_1; X^+ Z^+, t_3) = 1. \quad (44)$$

That is, given the same conditions as in (43) one can conclude that at the time t_2 the particle was in a spin state $S_z = 1/2$. This inference is not particularly surprising when one

remembers that, as shown in Sec. V A, the outcome of a measurement allows one to infer the state of the measured system prior to the time when the measurement takes place, if one uses an appropriate consistent family.

Note that there is no inconsistency between (43) and (44), because the conclusion $S_x = 1/2$ holds within framework \mathcal{F}_2 , and $S_z = 1/2$ within framework \mathcal{F}_3 , and these two frameworks are incompatible with each other, even though each one separately is compatible with \mathcal{F}_1 . Thus we have an example of the possibility noted in Sec. III B, of two logical arguments based upon the same assumption, but using incompatible frameworks, and whose conclusions cannot, therefore, be combined.

The correctness of $S_z = 1/2$, within the framework \mathcal{F}_3 , can be verified, in principle, by inserting a third device between the first two, Fig. 1, but this time of a type which measures S_z , and verifying that it yields the same result as the final device the particle passes through. One may worry that this third device somehow “creates” a value of S_z which was “not there” before the particle passed through it. Such a worry is best put to rest by means of a precise analysis, and the reader is invited to carry out the appropriate calculations using a framework \mathcal{F}_4 in which, with all three devices present, S_z is specified both at t_2 and at a time $t_{2.5}$ when the particle is between the two devices which measure S_z . It is easily shown that S_z has the same value at both these times; thus, passing through the intermediate device does not alter this component of the spin.

V C Double Slit

A complete discussion of the paradoxes associated with double-slit diffraction [26] would require, at the very least, a theory of quantum counterfactuals, and that is beyond the scope of the present article. A significant insight into the source of the conceptual difficulties can, nonetheless, be obtained using the tools of Secs. III and IV. In order to focus on the essentials, we shall use the idealization that at time t_1 the particle is described by a wavepacket $\psi_1(\mathbf{r})$ which is approaching the slits from the left in the geometry of Fig. 2, and that at a later time t_2 the wave packet resulting from $\psi_1(\mathbf{r})$ by unitary time evolution consists of three pieces, two of which are waves confined to the regions R_A and R_B just behind the two slits and moving to the right, and the third a reflected wave located to the left of, and traveling away from the slits. And we shall regard the presence, at time t_2 , of the particle in R_A , represented by a projector P_A , as equivalent to the statement that “the particle passed through slit A ”, and likewise its presence in R_B , projector P_B , as equivalent to its having passed through slit B . In addition, let the projector $P = P_A + P_B$ correspond to the region

$$R = R_A \cup R_B. \quad (45)$$

(The projector P_A acting on a wave function $\psi(\mathbf{r})$ produces a function $\tilde{\psi}(\mathbf{r})$ equal to $\psi(\mathbf{r})$ for \mathbf{r} inside R_A , and equal to 0 for \mathbf{r} outside R_A ; P_B is defined in a similar way.) By time t_3 the particle, if it has passed through the slit system at all, will have been detected by one of the detectors in the diffraction region, shown as circles in Fig. 2, where for convenience the distance between the detectors and the slits has been considerably shortened.

All the frameworks or consistent families in the discussion which follows will include an initial state

$$|\Psi_1\rangle = |\psi_1, D_1, D_2, \dots, D_m\rangle \quad (46)$$

at time t_1 , where D_j indicates that detector j is ready to detect a particle. The first family \mathcal{F}_1 we wish to consider has, in addition to the initial state, projectors D_1^*, D_2^*, \dots at t_3 , where D_j^* is a projector indicating that detector j has detected a particle. Since \mathcal{F}_1 involves simple histories at just two times, the consistency condition is satisfied, and the weights (26) may be used to calculate the probability

$$\Pr(D_j^*, t_3 \mid \Psi_1, t_1) \quad (47)$$

that, given the initial conditions, the j 'th counter will trigger. Its dependence on j will exhibit the usual diffraction pattern.

The next family of interest, \mathcal{F}_2 , is obtained by adding to the events of \mathcal{F}_1 an event corresponding to the particle being in the region R (45), projector P , at time t_2 . This is again a consistent family in which one can calculate quantities such as

$$\Pr(P, t_2 \mid \Psi_1, t_1) \quad (48)$$

the probability that the particle was not reflected by the slit system, and

$$\Pr(D_j^*, t_3 \mid P, t_2; \Psi_1, t_1), \quad (49)$$

the probability that the j 'th counter will trigger if the particle passes through the slit system. The latter shows the same diffraction pattern as (47), as a function of j , aside from a multiplicative constant. On the other hand, attempting to add to \mathcal{F}_2 the events that the particle is in one of the regions R_A or R_B at t_2 results in an inconsistent family, despite the fact that the corresponding projectors P_A and P_B can be included in a Boolean algebra of projectors for the tensor product \mathcal{H}^3 . Thus, while one can consistently say that “the particle was in R ” at time t_2 , it makes no sense, given this framework, to say that “the particle was in R_A , or it was in R_B ”. But given that R , (45), is the union of two disjoint regions R_A and R_B , it is difficult not to interpret the second phrase as equivalent to the first. Thus we have a situation where it may actually be helpful, in order to avoid confusion, to employ the projectors themselves in place of the corresponding English phrases.

The preceding discussion shows that the consistent history analysis supports the usual intuition that it is somehow improper to talk about which slit the particle passed through. However, it replaces an intuition which is always a little vague with a precise mathematical criterion based upon the Hilbert space structure of quantum mechanics. That this represents a significant advance is evident when one comes upon circumstances, such as that in which a detector is placed directly behind one of the slits, in which it seems intuitively plausible that one should be able to specify the slit through which the particle passed.

Even without displacing the detectors, one can construct a framework which permits one to say which slit the particle passed through, once again for convenience interpreting this as an assertion about its presence in R_A or R_B at t_2 . Let \mathcal{F}_3 be the family generated by P_A and P_B at t_2 along with Ψ_1 at t_1 , and (for the moment) no events at t_3 . As these simple histories involve only two times, consistency is automatic. One can then calculate the probabilities

$$\Pr(P_A, t_2 \mid \Psi_1, t_1), \Pr(P_B, t_2 \mid \Psi_1, t_1), \quad (50)$$

that the particle passed through slit A and slit B , respectively, and demonstrate that the particle surely did *not* pass through *both* slits by using the fact that $P_A P_B = \emptyset$. If, however,

one attempts to add to \mathcal{F}_3 events of the sort D_j^* at t_3 , the result is an inconsistent family. From this we see that what prevents one from sensibly talking about which slit the particle passes through in the usual discussions of double slit diffraction is directly tied to the (implicit) requirement of being able to describe the point at which the particle will arrive, or be detected, in the diffraction plane. A description which includes which slit the particle goes through can also consistently include events referring to the detectors at time t_3 ; these events must, however, be suitable MQS states. This suggests that the absence of a precise, clear discussion of two-slit diffraction in textbooks is not unrelated to the absence of a clear treatment of MQS states and the measurement problem. A consistent logical analysis using the concept of a framework is able to dispose of both problems simultaneously.

V D Emergence of the Classical World

Both Gell-Mann and Hartle [15], and Omnès [14] have discussed, from slightly different perspectives, how classical physics expressed in terms of suitable “hydrodynamic” variables emerges as an approximation to a fully quantum-mechanical description of the world when the latter is carried out using suitable frameworks, i.e., families of consistent histories. It is not our purpose to recapitulate or even summarize their detailed technical discussions, but instead to indicate the overall strategy, as viewed from the perspective of this paper, and how it is related to processes of “decoherence” [28] which arise when a particular quantum subsystem of interest interacts with a suitable environment, both of which are part of a larger (closed) quantum system.

The basic strategy of Gell-Mann and Hartle can be thought of as the search for a suitable “quasi-classical” framework, a consistent family whose Boolean algebra includes projectors appropriate for representing coarse-grained variables, such as average density and average momentum inside volume elements which are not too small, variables which can plausibly be thought of as the quantum counterparts of properties which enter into hydrodynamic and other descriptions of the world provided by classical physics. Hence it is necessary first to find suitable commuting projectors, and then to show that the consistency conditions are satisfied for the corresponding Boolean algebra. Omnès states his strategy in somewhat different terms which, however, seem at least roughly compatible with the point of view just expressed. (The actual technical calculations of Gell-Mann and Hartle, and of Omnès are based on simple, rather than generalized histories, in the terminology of Sec. IV B.)

One might worry that the strategies of Omnès, and Gell-Mann and Hartle, are incompatible with the formulation of quantum theory contained in the present paper, because their consistency conditions employ a density matrix, whereas that in Sec. IV C does not. However, the difference is probably of no great importance when discussing “quasi-classical” systems involving large numbers of particles, for the following reason. In classical statistical mechanics one knows (or at least believes!) that for macroscopic systems the choice of ensemble—microcanonical, canonical, or grand—is for many purposes unimportant, and, indeed, the average behavior of the ensemble will be quite close to that of a “typical” member. Stated in other words, the use of probability distributions is a convenience which is not “in principle” necessary. Presumably an analogous result holds for quantum systems of macroscopic size: the use of a density matrix, both as an “initial condition” and as part of the consistency requirement may be convenient, but it is not necessary when one is discussing

the behavior of a closed system. Of course, there are issues here which deserve serious study; the proper formulation of time-dependent phenomena in quantum statistical mechanics is an open problem, Sec. VI B.

The task of finding an appropriate quasi-classical consistent family is made somewhat easier by two facts. The first is that decoherence (as defined above) is quite effective in reducing “off-diagonal” terms $C(\tilde{M}^{(\alpha)}, \tilde{M}^{(\beta)})$, with $\alpha \neq \beta$, in the consistency functional (22) for a suitably chosen Boolean algebra representing quasi-classical variables in circumstances in which thermodynamic irreversibility plays a role. The second is that the consistency functional is a continuous function of its arguments, and hence it is plausible that by making small changes in the projectors forming the Boolean algebra, one can reduce the off-diagonal elements to zero, assuming they are already small (see Dowker and Kent [19]) so that the consistency condition (25) is satisfied. Since there is in any case some arbitrariness in choosing which quantum projectors to associate with particular coarse-grained hydrodynamic variables, small changes in these projectors are unimportant in terms of their physical interpretation. Thus exact consistency does not seem difficult to achieve “in principle”, even if in practice theoretical physicists are unlikely to be worried if the “off-diagonal” elements of the consistency functional are not precisely zero, as long as they are suitably small in comparison with the “diagonal” weights $C(\tilde{P}, \tilde{P})$ which enter into a calculation of probabilities.

There are many frameworks which are *not* quasi-classical, and from a fundamental point of view there is no reason to exclude using one of these to describe the world. The fundamental principles of quantum mechanics no more dictate the choice of a framework than the concepts of geometry dictate which projection a draftsman must use for representing a three-dimensional object. In both cases the issues are practical ones, related to what one is trying to achieve. As noted in Sec. V A, a framework which includes MQS states of a measurement apparatus after the measurement precludes any attempt to think of the process of its interaction with the measured system as a “measurement” in the ordinary sense of that word. Using MQS states to describe Schrödinger’s cat does nothing whatsoever to the cat, but it does make it impossible to discuss whether the cat is or is not alive, for such a discussion will employ concepts of a quasi-classical kind which cannot be included in the chosen framework. And any quantum description which wishes to make contact with the world of everyday experience or of experimental physics must employ some framework which allows a description of the appropriate kind.

As a final remark, the fact that in Sec. V A above no mention was made of decoherence in addressing the “measurement problem” should not be taken to mean that considerations of decoherence are irrelevant to discussions of quantum measurements; quite the opposite is the case. For example, the fact that certain physical properties, such as pointer positions in a properly designed apparatus, have a certain stability in the course of time despite perturbations from a random environment, while other physical properties do not, is a matter of both theoretical and practical interest. However, decoherence by itself cannot single out a particular framework of consistent histories, nor can it disentangle conceptual dilemmas brought about by mixing descriptions from incompatible frameworks. To the extent that the latter form the heart of the “measurement problem”, decoherence will not resolve it, and claims to the contrary merely add to the confusion.

VI Summary and Open Questions

VI A Summary

The description of a quantum system and the reasoning process relating one description to another are best thought of in terms of a *framework*, which serves to establish, in effect, a topic of conversation or a point of view, and *statements*, which are assertions about the state of a quantum mechanical system which are either true or false within the framework with which they are associated. A classical analogy, Sec. II A, in which a framework must be chosen by the person constructing the description, and truth is relative to this framework, suggests that there is no reason to treat quantum descriptions as less “objective” than their classical counterparts. The most important difference between classical and quantum descriptions is the fact that in the quantum case there are *incompatible* frameworks, any one of which could be employed for constructing descriptions of a given system, but which cannot be combined, either for the process of describing the system or reasoning about it. This incompatibility has no direct classical analog, and it leads to the rule which states that any description of a quantum system must either employ a single framework or a compatible family of frameworks. Similarly, a logical argument, leading from certain descriptions, taken as assumptions, to other descriptions which form the conclusions, must use a single framework or a compatible family of frameworks. In the case of a classical mechanical system, a single framework suffices for all descriptions, and this framework is often chosen implicitly. A quantum framework can also be chosen implicitly, but carelessness can lead to invalid reasoning and to paradoxes.

The same combination of framework and statement can be used to describe the behavior of a quantum system as a function of time. In this case the framework is usually called a “consistent family of histories”. Once again, a valid description or history must be based upon a single framework or consistent family, or a compatible set of consistent families, and a logical argument leading from assumptions to conclusions must likewise employ a single consistent family or a compatible set of families. It is convenient to represent (generalized) histories involving n times using projectors on the tensor product of n copies of the Hilbert space. The consistency condition can then be expressed using a bilinear *consistency functional* of such projectors, and when consistency is satisfied, the weights which determine conditional probabilities can also be computed using this consistency functional.

Applications of the formalism just described include measuring processes, the state of a quantum system between measurements, double-slit diffraction, and the emergence of the classical world from a fully quantum mechanical theory.

VI B Open Questions

The discussion of consistency in Sec. IV C was carried out assuming a finite-dimensional Hilbert space in order to ensure, among other things, that the trace (22) and the minimal elements (23) of the Boolean algebra $\tilde{\mathcal{B}}$ exist. It seems likely that by making suitable restrictions on $\tilde{\mathcal{B}}$, the concept of consistency can be extended in a satisfactory way to an infinite-dimensional Hilbert space, but this needs to be worked out in detail. There are similar technical issues associated with frameworks which employ an infinite collection of

elementary statements, the compatibility of infinite collections of frameworks, and histories defined at an infinite set of times.

The structure for quantum reasoning proposed in this paper includes no mechanism for dealing with counterfactuals (“if the counter had not been located directly behind the slit, then the particle would have ...”). Inasmuch as many quantum paradoxes, including some of the ones associated with double-slit diffraction, and certain derivations of Bell’s inequality and analogous results, make use of counterfactuals, disentangling them requires an analysis which goes beyond what is found in the present paper. As philosophers have yet to reach general agreement on a satisfactory scheme for counterfactual reasoning applied to the classical world [27], an extension which covers all of quantum reasoning is likely to be difficult. On the other hand, a scheme sufficient to handle the special sorts of counterfactual reasoning found in common quantum paradoxes probably represents a simpler problem.

The proper formulation of time-dependent phenomena in quantum statistical mechanics remains an open question. On the one hand, as noted in Sec. V D, it seems plausible that various formulas which yield the “average” behavior of some macroscopic system are not in need of revision, since, among other things, this “average” is likely to be essentially the same as the behavior of a typical member of an ensemble, and because decoherence (interaction with the environment) is very effective in removing violations of the consistency conditions if one chooses an appropriate “quasi-classical” consistent family. The results obtained in [14, 15] are quite encouraging in this connection. On the other hand, thought needs to be given to those cases in which different members of an ensemble have significantly different behavior, that is, where “fluctuations” are important. In any event, it would be useful to have a restatement of the fundamental principles of quantum statistical mechanics based upon a fully consistent interpretation of standard quantum mechanics.

Can the structure of reasoning developed in this paper for non-relativistic quantum mechanics be extended to relativistic quantum mechanics and quantum field theory? Various examples suggest that the sort of peculiar non-locality which is often thought to arise from violations of Bell’s inequality and various EPR paradoxes will disappear when one enforces the rule that consistent quantum arguments must employ a single framework. While this is encouraging, it is also true that locality (or the lack thereof) in non-relativistic quantum theory has not yet been carefully analyzed from the perspective presented in this paper, and hence must be considered among the open questions. And, of course, getting rid of spurious non-localities is only a small step along the way towards a fully relativistic theory.

Even in the domain of non-relativistic (and non-counterfactual) quantum reasoning, there is no proof that the scheme presented in this paper is appropriate and adequate for all of standard quantum mechanics. Indeed, standard quantum mechanics contains a great deal of seat-of-the-pants intuition which has never been formalized, and thus it is hard to think of any way of testing the system of reasoning presented here apart from applying it to a large number of examples, to see if it yields what the experts agree are the “right answers”, and what the experimentalists find in their laboratories. During the past ten years consistent history ideas have been applied to many different situations without encountering serious problems, but there is always the possibility that the fatal flaw lies just around the next corner. If such a flaw exists, it should not be hard to identify, since the scheme of reasoning presented in this paper has precise rules and is based on “straight” quantum mechanics, without hidden variables, adjustable parameters, modifications of the Schrödinger equation,

and excuses of the “for all practical purposes” type. The reader who considers it defective is invited to point out the problems!

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Figure Captions

Figure 1: Spin 1/2 particle passing through devices which measure S_x and S_z . A third device may be added at the position shown by the dashed line.

Figure 2: Diffraction of a particle by two slits (see text).

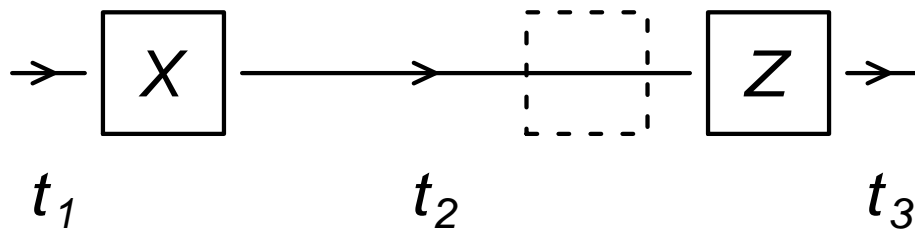


Figure 1.

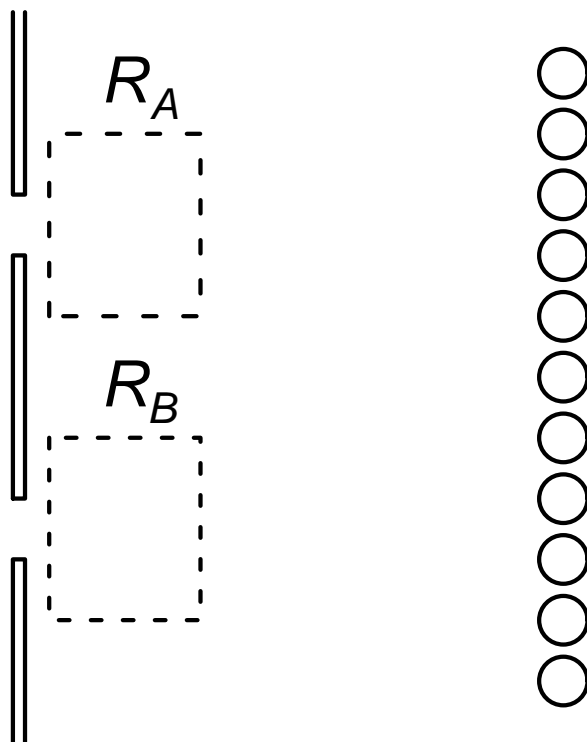


Figure 2.